SECTION 11.1: TAYLOR POLYNOMIALS

In this section, we aim to approximate functions using polynomials. We know differentiable functions can be approximated by linear functions (their tangent lines), and in this section we generalize that concept.

Suppose f is continuous at $x = a$ and we wish to approximate f near $x = a$ by a constant function, $f(x) \approx b$. Since f is continuous, as $x \to a$, $f(x) \to f(a)$, so we have that $f(a) = b$ so, unsurprisingly, we write $f(x) \approx f(a)$.

Suppose f' is continuous at $x = a$ and we wish to approximate f near $x = a$ by a linear function, let's say $f(x) \approx m(x-a)+b$. Since f is continuous, as $x \to a$, $f(x) \to f(a)$. This gives us that $f(a) = m(a-a)+b = b$, so $f(a) = b$. Since f is differentiable, we can insist that $f'(x) \approx D_x\,[m(x-a)+b] = m$. Since f' is continuous, we have that as $x \to a$, $f'(x) \to f'(a)$, so that $f'(a) = m$. Hence, we have recovered the linear approximation (tangent line) formula: $f(x) \approx f'(a)(x-a) + f(a)$.

Suppose f'' is continuous at $x = a$ and we wish to approximate f near $x = a$ by a quadratic function, let's say $f(x) \approx A(x-a)^2 + B(x-a) + C$. Once again, since f is continuous, as $x \to a$, $f(x) \to f(a)$, so we have that $f(a) = A(a-a)^2 + B(a-a) + C$, so, $f(a) = C$.

Since f is differentiable, we insist $f'(x) \approx D_x\left[A(x-a)^2 + B(x-a) + C\right] = 2A(x-a) + B$. Since $f''(a)$ exists, f' is continuous at $x = a$ so that as $x \to a$, $f'(x) \to f'(a)$. Hence, we get $f'(a) = 2A(a-a) + B = B$.

Since f is twice-differentiable, we insist $f''(x) \approx D_x^2\left[A(x-a)^2 + B(x-a) + C\right] = 2A$. Since f'' is continuous at $x = a$, as $x \to a$, $f''(x) \to f''(a)$ so $f''(a) = 2A$ or $A = \frac{1}{2}f''(a)$.

Putting all this together we get that 'near' $x = a$: $f(x) \approx \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a)$.

Repeating this process, we find:

$$f(x) \approx \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a)$$

and

$$f(x) \approx \frac{1}{24}f^{(4)}(a)(x-a)^4 + \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a)$$

If f as n derivatives, we can form an nth degree polynomial using the following recipe:

DEFINITION: The nth degree Taylor Polynomial for a function f **centered** at $x = a$ is:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{24}f^{(4)}(a)(x-a)^4 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

If we employ the convention that $f^{(0)}(a) = f(a)$, we can write the formula more compactly as

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

EXAMPLE 1: Find the 4th degree Taylor Polynomial for the given function at the given center:

1. $f(x) = \sin(x);\ a = 0$

Ans: $\sin(x) \approx x - \dfrac{x^3}{6}$

2. $f(x) = \cos(x);\ a = 0$

Ans: $\cos(x) \approx 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$

3. $f(x) = \ln(x);\ a = 1$

Ans: $\ln(x) \approx (x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{3}(x-1)^3 - \dfrac{1}{4}(x-1)^4$

4. $f(x) = \dfrac{1}{x};\ a = 1$

Ans: $\dfrac{1}{x} \approx 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$

We may well wonder 'how good' these polynomial approximations are. To that end, we define the remainder as follows: $r_n(x) = f(x) - p_n(x)$. That is, $r_n(x)$ measures the difference between the actual function value $f(x)$ and the approximate value $p_n(x)$. We have the following theorem:

REMAINDER THEOREM: If f has $n+1$ derivatives in an open interval containing a and x, then there is some value c between a and x such that:

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

NOTE 1: Does this remind you at all of the remainder theorem for Alternating Series?

NOTE 2: When $n = 0$, $p_0(x) = f(a)$, so this theorem says there is some value c between a and x such that

$$f(x) - f(a) = f'(c)(x-a) \qquad \Longleftrightarrow \qquad f'(x) = \frac{f(x) - f(a)}{x - a}$$

This is none other than the MVT from Calculus.[1]

The Remainder Theorem is an example of an 'existence' theorem it tells you a value c exists, but doesn't give you any means to find it. Like the remainder theorems we learned in the previous chapter, we use this theorem to **bound** the error associated with a given approximation.

[1] In Exercise 82, you can walk through the proof of the remainder theorem which uses the integral form of the MVT!

EXAMPLE 2: Use $p_4(x)$ from Example 1 to approximate $f(x)$. Estimate the error using the remainder theorem.

1. $f(x) = \sin(x)$; $x = 1$

Ans: $\sin(1) \approx 0.8\overline{3}$; error estimate: $0.008\overline{3}$

2. $f(x) = \ln(x)$; $x = 2$

Ans: $\ln(2) \approx 0.58\overline{3}$; error estimate: 0.2

EXAMPLE 3: Let $p_n(x)$ be the nth degree Taylor Polynomial for $f(x) = \ln(x)$ centered at $x = 1$.

1. Use the remainder theorem to find n so $p_n(2) \approx \ln(2)$ with an error of at most ± 0.001.

Ans: $n = 999$

2. Find $p_n(2)$ using the n you found in part 1.

Ans: $p_{999}(2) = 0.6936\ldots$

HOMEWORK: Section 11.1: 9 - 75, 82* odd.